10^{th} Exercise sheet for Advanced Quantum Mechanics in WS16

Exercise 22 *Harmonic oscilator with time-independent perturbation Hamiltonian*

Consider a one-dimensional harmonic oscillator with mass m and frequency ω, which is perturbed by the time-independent Hamiltonian W.

a) Let $W = \alpha Q$, where Q denotes the position operator. Calculate the corrections to the unperturbed energies $E_i^{(0)}$ of the harmonic oscillator up to order $\mathcal{O}(\alpha^2)$ and determine the corrections to the eigenstates to order $\mathcal{O}(\alpha)$. Compare the results with the exact solutions in this case.

b) Now let $W = \beta Q^2$ and calculate the first-order corrections to the unperturbed energies. Again, compare the results with the exact solutions.

c) Determine the all non-vanishing matrix elements due to a perturbation potential $W = \gamma Q^3$ and calculate the energy corrections to second order, i.e. $\mathcal{O}(\gamma^2)$. Furthermore, determine the distances of two neighbouring energy levels including the perturbation Hamiltonian.

Exercise 23 *Two-atom molecule excitations due to electromagnetic pulse*

The low-frequency excitations of a two-atom molecule can be approximately described by a charged harmonic oscillator $V(x) = m\omega_0^2 x^2/2$ with mass m, frequency ω_0, charge q, and eigenstates $|n\rangle$. During the time $0 \leq t \leq \tau$ the molecule is exposed to an electromagnetic "pulse" with amplitude $\mathcal{E}$, which is described by the potential

$$W(t) = -q\mathcal{E}x\theta(t)\theta(\tau - t), \tag{1}$$

where $\theta(x)$ denotes the Heavyside step function ($\theta(x > 0) = 1$, $\theta(x < 0) = 0$). For $t < 0$ the system shall be in the ground-state $|0\rangle$.

a) Calculate the probability $P_{10}(\tau)$ for the transition from $|0\rangle$ to $|1\rangle$ in first-order perturbation theory; use creation and annihilation operators. Sketch $P_{10}(\tau)$ as a function of τ.

b) Now consider the transition $|0\rangle \to |2\rangle$. Show that for the determination of $P_{20}(\tau)$ one needs at least second-order perturbation theory. Using the formula

$$a_{\text{fi}}^{(2)}(t) = \left(\frac{-\mathrm{i}}{\hbar}\right)^2 \sum_{\text{m}} \int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t'' e^{\mathrm{i}\omega_{\text{fm}}t'} W_{\text{fm}}(t') e^{\mathrm{i}\omega_{\text{mi}}t''} W_{\text{mi}}(t''), \tag{2}$$

calculate $P_{20}(\tau)$, where $a_{\text{fi}}^{(2)}(t)$ is the second-order transition amplitude for $|\text{i}\rangle \to |\text{f}\rangle$.