9^{th} Exercise sheet for Advanced Quantum Mechanics in WS16

Time dependent perturbation theory and variational method

Exercise 20 *Damped harmonic oscillator*

The motion of a particle with mass m shall be described by the time-dependent Hamiltonian H,

$$H = \frac{p^2}{2m} e^{-\alpha t} + V(x) e^{\alpha t}, \tag{1}$$

where α is a positive constant.

a) Study the classical motion of the particle; show that it is subject to two forces, one conservative, due to the potential $V(x)$, and the other one a damping force.

b) Consider the case of an harmonic potential,

$$V(x) = \frac{1}{2} m \omega^2 x^2, \tag{2}$$

i.e. H then describes a damped harmonic oscillator. If $\alpha \ll \omega$, for $\alpha t \ll 1$ we can use perturbation theory. Assume at $t = 0$ the harmonic oscillator is in the ground state. At first order in α, determine which states can be excited, and calculate the transition probability into the corresponding excited states as a function of t.

c) Find the general solution of the Schrödinger equation for $V(x) = 0$.

d) Using the solution of c), and assuming at $t = 0$ the wave function is

$$u(x) = A \exp\left\{ -\frac{x^2}{a^2} + \mathrm{i}\beta x \right\}, \tag{3}$$

determine the wave function for $t \to \infty$.

Exercise 21 *Helium atom*

Consider the Helium atom, or more generally a $Z-2$ times ionized atom as e.g. Li^+, Be^{++}, etc. with Hamiltonian

$$H=\sum_{i=1,2}\frac{\boldsymbol{p}_i^2}{2m}-\frac{Ze^2}{4\pi\epsilon_0}\frac{1}{r_1}-\frac{Ze^2}{4\pi\epsilon_0}\frac{1}{r_2}+\frac{e^2}{4\pi\epsilon_0}\frac{1}{r_{12}}, \tag{4}$$

where $r_1=|\boldsymbol{r}_1|$, $r_2=|\boldsymbol{r}_2|$, and $r_{12}=|\boldsymbol{r}_1-\boldsymbol{r}_2|$. In the ground state the total spin $\boldsymbol{S}=\boldsymbol{S}_1+\boldsymbol{S}_2$ of both electrons is zero. The wave function of the ground state Ψ can be decomposed into a symmetric orbital part Φ and into a antisymmetric spin part χ

$$\Psi(\boldsymbol{r}_1,S_{z1};\boldsymbol{r}_2,S_{z2})=\Phi(\boldsymbol{r}_1,\boldsymbol{r}_2)\,\chi_0^0(S_{z1},S_{z2}), \tag{5}$$

where

$$|\chi_0^0\rangle=\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle\right), \tag{6}$$

as in exercise 16. The goal of this exercise is to find an approximation of the energy of the ground state due to the variational method.

a) For the orbital part Φ we assume a trial wave function of the form

$$\Phi_\alpha(\boldsymbol{r}_1,\boldsymbol{r}_2)=\frac{\alpha^3}{\pi}e^{-\alpha r_1-\alpha r_2}, \tag{7}$$

which depends on a still unknown parameter $\alpha>0$. Prove that the wave function is already normalized.

b) Calculate the energy eigenvalue $\langle\Phi_\alpha|H|\Phi_\alpha\rangle$ as a function of α and determine its minimum. Express $\alpha_{\min}$ with the Bohr radius

$$a_0=\frac{\hbar^2}{m}\frac{4\pi\epsilon_0}{e^2}. \tag{8}$$