3rd Exercise sheet for Advanced Quantum Mechanics in WS16
Spin and Rotations

Exercise 8 *Spin 1 Hamiltonian*

The Hamiltonian for a spin 1 system is given by

$$H = AS_z^2 + B(S_x^2 - S_y^2) \tag{1}$$

Solve this problem exactly to find the normalized energy eigenstates and eigenvalues. Is this Hamiltonian invariant under time reversal? How do the normalized eigenstates transform under time reversal?

Exercise 9 *Transformation behaviour of operators under rotation*

a) Let A and B be operators and x a parameter. Show that

$$e^{xA}Be^{-xA} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \underbrace{[A,[A,\ldots[A}_{n\times},B\underbrace{]]\cdots]}_{n\times}. \tag{2}$$

Hint: Show that both $F_{\text{LHS}}(x) = e^{xA}Be^{-xA}$ and $F_{\text{RHS}}(x) = \sum_{n=0}^{\infty}\cdots$ solve the differential equation $\frac{\mathrm{d}}{\mathrm{d}x}F = [A,F]$ and explicitly show the equality in Eq. (2) for a simple value of x. Then argue why that implies the equality in Eq. (2).

b) Let V_k, with $k = 1,2,3$, be the cartesian components of a vector operator $\mathbf{V}$, i.e. the commutators of its components with the angular momentum operator $\mathbf{J}$ shall satisfy

$$[J_i, V_j] = \mathrm{i}\epsilon_{ijk}V_k. \tag{3}$$

A rotation by an angle α around a normalized axis $\mathbf{n}$ transforms $\mathbf{V}$ to:

$$\mathbf{V}' = e^{-\mathrm{i}\alpha\mathbf{n}\cdot\mathbf{J}}\mathbf{V}e^{\mathrm{i}\alpha\mathbf{n}\cdot\mathbf{J}}. \tag{4}$$

Derive an expression for $\mathbf{V}'$ not containing the angular momentum operator $\mathbf{J}$.
Hint: Show by induction that $[\mathrm{i}\mathbf{n}\cdot\mathbf{J}, K_m\mathbf{V}] = K_{m+1}\mathbf{V}$ for vector operators $\mathbf{V}$ where the action of K_m is to apply the cross product with $\mathbf{n}$ m times, e.g. $K_0\mathbf{V} = \mathbf{V}$, $K_1\mathbf{V} = \mathbf{n}\times\mathbf{V}$, $K_2\mathbf{V} = \mathbf{n}\times(\mathbf{n}\times\mathbf{V})$, and so on.

c) Now consider the matrix elements of a rotation around the y-axis:

$$d^{(j)}_{mm'}(\beta) = \langle j,m|e^{-\mathrm{i}\beta J_y}|j,m'\rangle. \tag{5}$$

Calculate the following sum for arbitrary (integer/half-integer) spin j:

$$\sum_{m=-j}^{j} m\left|d^{(j)}_{mm'}(\beta)\right|^2 \tag{6}$$

d) For arbitrary j, show that

$$\sum_{m=-j}^{j} m^2\left|d^{(j)}_{mm'}(\beta)\right|^2 = \frac{1}{2}j(j+1)\sin^2\beta + m'^2\frac{1}{2}(3\cos^2\beta - 1). \tag{7}$$