1^{st} Exercise sheet for Advanced Quantum Mechanics in WS16

Recapitulation of Quantum Mechanics

Exercise 1 *Potential well*

Consider a stationary quantum mechanical system described by the following potential

$$V(x) = \begin{cases} \infty & x < 0\,, \\ -V_0 & 0 \leq x \leq a\,, \\ 0 & x > a\,. \end{cases} \tag{1}$$

and derive the quantization condition for bounded states:

1. Find the solutions of the time-independent Schrödinger equation in all three regions.
2. Derive the quantization condition from regularity and continuity conditions of the wave function.
3. Determine the ground state and thereof conclude the condition for the minimal depth of the potential well V_0^{min}.

Exercise 2 *Harmonic oscillator*

The Hamiltonian of a one-dimensional harmonic oscillator is given by

$$H(x,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2, \tag{2}$$

where the momentum operator p and position operator x satisfy the usual commutator relation $[x,p] = \mathrm{i}\hbar$. To eliminate the quantities $\hbar$, m, and ω one introduces the rescaled operators P and X,

$$P = \frac{1}{\sqrt{m\hbar\omega}}p, \qquad Q = \sqrt{\frac{m\omega}{\hbar}}x, \tag{3}$$

so that $[Q,P] = \mathrm{i}$ and the energy is given in units of $\hbar\omega$, i.e. $H = \hbar\omega H'$. The Schrödinger equation of the Hamiltonian $H'(X,P)$ then reads

$$H'|\psi\rangle = \frac{1}{2}\left(P^2 + Q^2\right)|\psi\rangle = \epsilon|\psi\rangle, \tag{4}$$

where ϵ denotes the eigenvalue of H'.

1. For $n \in \mathbb{N}_0$ show that

$$\frac{1}{2}\left(P^2 + Q^2\right)\left(Q \pm \mathrm{i}P\right)^n|\psi\rangle = \left(\epsilon \mp n\right)\left(Q \pm \mathrm{i}P\right)^n|\psi\rangle, \tag{5}$$

   and explain the meaning of this relation.

2. Find normalized eigenfunctions and the corresponding eigenvalues of the Hamiltonian $H'(X, P)$.

3. Calculate all commutators of the operators

$$a = \frac{1}{\sqrt{2}}\,(Q + \mathrm{i}P)\,, \qquad a^\dagger = \frac{1}{\sqrt{2}}\,(Q - \mathrm{i}P)\,. \tag{6}$$

   Express the wave function of the n-th excited state by the wave function of the ground state and the operators a and $a^\dagger$.

4. Calculate the (infinite-dimensional) matrix representation of P and Q using the energy eigenstates.

Exercise 3 *Angular momentum operator*

a) In quantum mechanics rotations around an axis $\mathbf{n}$ by an angle ϕ are described by an operator

$$U(\phi) = \exp\left[-(\mathrm{i}/\hbar)\phi\,\mathbf{n}\cdot\mathbf{J}\,\right], \tag{7}$$

where $\mathbf{J}$ is the angular momentum operator of the system. Show that a rotation by an angle of 2π is the identity operator (its negative) for integer spins (half-integer spins), i.e. show that the following relation holds:

$$U(\phi)|_{\phi=2\pi} = (-1)^{2j}. \tag{8}$$

b) In the special case $j = 1$ we have the relation $\mathbf{J} = \hbar\mathbf{I}$, where $\mathbf{I}$ fulfils the relation $(\mathbf{n}\cdot\mathbf{I})^3 = \mathbf{n}\cdot\mathbf{I}$. Show that $U(\phi)$ can be written in the form

$$U(\phi)|_{j=1} = a_0 + a_1\mathbf{n}\cdot\mathbf{I} + a_2(\mathbf{n}\cdot\mathbf{I})^2, \tag{9}$$

with $a_0, a_1, a_2 \in \mathbb{C}$. Determine these constants.